DIFFERENTIAL EQUATIONS

Points to be covered in this topic

1. INTRODUCTION
2. ORDER AND DEGREE
3. EQUATIONS IN SEPARABLE FORM
4. HOMOGENEOUS EQUATIONS
5. LINEAR DIFFERENTIAL EQUATIONS
6. EXACT EQUATIONS
7. APPLICATION IN SOLVING PHARMACOKINETIC EQUATIONS

❑ INTRODUCTION

- A differential equations that involves independent, dependent vaiables and derivatives of dependent variables.
- The following are the example of differential equations:

$$\frac{dy}{dx}+2y=0,\ \frac{d^2y}{dx^2}+3\frac{dy}{dx}+2=0$$

❑ ORDER & DEGREE

- Order of a differential equation: The order of a differential equation is the order of the highest derivative occurring in it.

The order of the diffrential equation $\left(\frac{d^3y}{dx^3}\right)^{1/2}=\left[1+\frac{dx}{dy}+\left(\frac{dy}{dx}\right)^2\right]^{2/3}$ is 3,

which is the order of $\left(\frac{d^3y}{dx^3}\right)$.

- **Degree of a differential equation: The degree of a differential equation is the highest power of the highest order derivative occurring in it after the equation is freed from radical signs and fractions in the derivatives**

The degree of the diffrential equation $5\left(\frac{d^2y}{dx^2}\right)^4 - \left(\frac{dy}{dx}\right)^2 + 2y = \sin x$ i 95.

since the highest order derivative is $\left(\frac{d^2y}{dx^2}\right)$ and its power is 4.

❑ EQUATION IN SEPARABLE FORM

- The general form of a differential equation is f(x) dx + ø(y) dy = 0, or equation which can be expressed that coefficient of dx is a function of x only and the coefficient of dy is a function of y only.
- Stepwise procedure to solve the equation in separable form
- Step I: Write the given differential equation in the form f(x) dx + ø(y) dy = 0
- Step II: Integrate both the sides and add an arbitrary constant on one side.

Example :1 $\frac{dy}{dx} = 3x^2 + \frac{1}{x}$

solution (i)we have, $\frac{dy}{dx} = 3x^2 + \frac{1}{x}$

This eqution can be witten in seprable from as:

$$dy=\left(3x^2 + \frac{1}{2}\right)dx$$

Integrate bot the sides,we get

$$f\ dy=\left(3x^2 + \frac{1}{2}\right)dx+c$$

$$y=f\ x^2dx + f\frac{1}{x}dx+c$$

$$y=\frac{3x^3}{3} + \log x+c$$

or $$y=x^3 + \log x+c$$

we have, $\frac{dy}{dx} + 2x = 3e^x$

$$\Rightarrow dy=\left(3e^x - 2x\right)dx$$

integrate, we get

$$f\ dy=f\left(3e^x - 2x\right)dx$$

$$y=3e^2 - x^2 + c$$

we have, $\frac{dy}{dx}$-cos(x+y)

put x+y-z

$1+\frac{dy}{dx}=\frac{dz}{dx}$

$\frac{dy}{dx}=\frac{dz}{dx}-1$

Equation (1)reduced to,

$\frac{dz}{dx}-1=\cos$

OR $\frac{dz}{dx}=1+\cos z \Rightarrow \frac{dz}{1+\cos z}=dx$

Intigrate, $f\frac{dz}{1+\cos z}f\ dx$

$f\frac{dz}{2\cos^2\frac{z}{2}}=x+c$

Or $\frac{1}{2}f\ \sec^2\left(\frac{z}{2}\right)dz=x+c$

$\frac{1}{2}2.\tan\left(\frac{z}{2}\right)=x+c$

OR $\tan\frac{z}{2}=x+c$

OR $\tan\left(\frac{x+y}{2}\right)=x+c$

we have , $2xy\frac{dy}{dx}=x^2+3y^2$

solution (ii) $2xy\frac{dy}{dx}=x^2 3y^2$

$\Rightarrow \frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$[Homogeneos]

put y - vx and $\frac{dy}{dx}$-v+x$\frac{dv}{dx}$

from eqution (1),we get

$v+x\frac{dv}{dx}=\frac{x^2 3v^2 x^2}{2x\ vx}=\frac{1+3v^2}{2v}$

$\Rightarrow x+\frac{dv}{dx}=\frac{1+3v^2}{2v}-\frac{v}{1}=\frac{1+3v^2-2v^2}{2v}=\frac{1+v^2}{2v}$

$\Rightarrow \frac{2v\ dv}{1+v^2}=\frac{dx}{x}$

Intigrate, $f\frac{2v\ dv}{1+v^2}=f\frac{dx}{x}$

$\log|1+v^2|=\log|x|+\log|c|$

$1+v^2=cx$

put $v=\frac{v}{x}$

$x^2+y^2=cx^2.x$ OR $x^2+y^2=cx^3$

$(xy^2+x)dx+(yx^2+y)dx=0.$

solution: we have ,

$(xy^2+x)dx+(yx^2+y)dy=0$

$\Rightarrow x(y^2+1)dx+y(x^2+1)dy=0$

Divinding by $(1+x^2)(1+y^2)$, we get

$\frac{xdx}{1+x^2}+\frac{y}{1+y^2}dy=0$

OR

Integrate,

$\int\frac{2x\,dx}{1+x^2}+\frac{2y\,dy}{\sqrt{1+y^2}}=\log|c|$ [log |c| is arbitarary constant]

$\log|1+x^2|+\log|1+y^2|=\log|c|$

or $\log|(1+x^2)(1+y^2)|=\log|c|$

$\Rightarrow(1+x^2)(1+y^2)=c$

We have, $\frac{dy}{dx}=y\tan 2x$

$\Rightarrow \frac{dy}{y}=\tan 2x\,dx$

Intigrate, $\int\frac{dy}{y}=\int\tan 2x\,dx$

$\log|y|-\frac{1}{2}\log|\sec 2x|=\log|c|$

$2\log|y|-\log|\sec 2x|=2\log|c|$

$\Rightarrow y^2\cos 2x = c^2$

$\Rightarrow y = c\sqrt{\sec 2x}$

put x=0, y=2, $2=c\sqrt{\sec 0}\Rightarrow 2=c$

putting the value of in eqution (2),we get

$y - 2\sqrt{\sec 2x}$ or $y^2\cos 2x\text{-}4$

❑ HOMOGENEOUS EQUATIONS

- A homologus equation of the form $\frac{dy}{dx}=\frac{x^2+xy}{2xy}$, where f(x,y) and g(x,y) are homogenous function in x and y of the same degree is said to be homogeneous differentiated equation.

$\frac{dy}{dx}=\frac{x^2+xy}{2xy}$ is a homogeneos differential eqution.

Stepwise procedure to solve a homogeneos differential equation:

Consider a requation $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$.

step-I:put y=vx, so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

step-II:Separating the variables.

step-III:Integrate

step-iv:put $v=\frac{y}{x}$ and simpalify

$$x^2\frac{dy}{dx}=x^2-2y^2+xy.$$

Solution (i) we have,

$$x^2\frac{dy}{dx}=x^2-2y^2+xy$$

OR $\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2}$ [Homogeneous]

put y=vx,so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Eqution (1)reduced to

$$v+x\frac{dv}{dx}=\frac{x^2-2v^2x^2+xvx}{x^2}=\frac{1-2v^2+v}{v}$$

$$\Rightarrow x\frac{dv}{dx}=1-2v^2$$

$$\Rightarrow \frac{dv}{1-2v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{dv}{1-\left(\sqrt{2v}\right)^2}=\frac{dx}{x}$$

Intigrate, $f\frac{dv}{1-\left(\sqrt{2v}\right)^2}=f\frac{dx}{x}$

$$\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2v}}{1-\sqrt{2v}}\right|=\log|x|+\log|c|$$

$$\frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2y}}{1-\sqrt{2y}}\right|=\log|xc|$$

OR $\quad LOG\left|\frac{X+\sqrt{2y}}{x-\sqrt{2y}}\right|=\log\left|(xc)^{2\sqrt{2}}\right|$

$$x+\sqrt{2y}=\left(x-\sqrt{2y}\right).(xy)^{2\sqrt{2}}$$

❑ LINER DIFFRENTIAL EQUATION

- A differential equation is said to be liner if the dependent variable and all of its derivates occur only in the first degree and not multiplied together.

$(y \log y)dx + (x - \log\ y)dy = 0$

Solution (ii) we have, $(y \log y)dx + (x - \log y)dy = 0$

Thish eqution can be written as

$$\frac{dy}{dx} = \frac{y \log y}{\log y - x}$$

OR $\frac{dx}{dy} = \frac{(\log y\text{-}x)}{y \log y} = \frac{\log y}{y \log y} - \frac{x}{y \log y}$

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

This is a linear differential equation in x.

$\therefore$ I.F.$= e^{\int p\,dy} = e^{\int \frac{1}{y \log y} dy}$

I.F.$= e^{\int \frac{1}{y \log y} dy}$ put log y=t, $\frac{1}{y}$

$= e^{\int \frac{1}{t} dt}$

$= e^{\int \log|t|} = t = \log y$

solution of (1)is given by

$x.\log y = \int \frac{1}{y} \log y\ dy + c$

$= \frac{(\log y)^2}{2} + c$

OR $x = \frac{\log y}{2} + \frac{c}{\log y}$

$(x+y+1)\frac{dy}{dx}=1.$

solution (i) we have ,

$(x+y+1)\frac{dy}{dx}=1$

OR $\frac{dx}{dy}=x+y+1$

OR $\frac{dx}{dy}-x=y+1$

which is linear differential eqution in x.

Here, P=1, Q=(Y+1)

$\therefore$ I.F. $=e^{\int -1.dy}=e^{-y}$

solution of (1) *is*

$x.\,e^{-y}=\int(y+1)e^{-y}dy+c \Rightarrow xe^{-y}=-y(y+1)e^{-y}-1e^{y}+c$

OR $x=(y+2)+ce^{y}$

$$x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}=0$$

solution (i) $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}$

OR $\left(x-\frac{y}{x^2+y^2}\right)dx+\left(y+\frac{x}{x^2+y^2}\right)dy=0$

Here, $M=x-\frac{y}{x^2+y^2} \Rightarrow \frac{\partial M}{\partial y}=0-\left[\frac{x^2+y^2-y.2y}{(x^2+y^2)^2}\right]=\frac{y^2-x^2}{(x^2+y^2)^2}$

and $N=y+\frac{x}{x^2+y^2}$

$\Rightarrow \frac{\partial N}{\partial x}=0+\left[\frac{x^2y^2-x.2x}{(x^2+y^2)}\right]$

$=\frac{y^2-x^2}{(x^2+y^2)^2}$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

∴ Eqution (1) is exact and its solution is given by

$$\int\left(x-\frac{y}{x^2+y^2}\right)dx + \int y\ dy = c$$

(treating y as constant) (terms free x in N)

$$\frac{x^2}{2}-Y.\frac{1}{y}\tan^{-1}\frac{1}{y}+\frac{y^2}{2}=0$$

$$x^2+y^2-2\tan^{-1}\frac{x}{y}=2c$$

OR $x^2+y^2=2\tan^{-1}\frac{x}{y}+c_1[\text{where } c_1=2c]$

❑ EXACT DIFFRENTIAL EQUATION

- If M and N are function of x and y, the equation M(x, y) dx + N(x,y) dy = 0 or it may be written as M dx + N dy = 0 is called exact when there exist a function f(x, y) of x and y such that d [f(x, y)] = M dx + N dy

$$x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}=0$$

solution (i) $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}$

OR $\left(x-\frac{y}{x^2+y^2}\right)dx+\left(y+\frac{x}{x^2+y^2}\right)dy=0$

Here, $M=x-\frac{y}{x^2+y^2} \Rightarrow \frac{\partial M}{\partial y}=0-\left[\frac{x^2+y^2-y.2y}{(x^2+y^2)^2}\right]=\frac{y^2-x^2}{(x^2+y^2)^2}$

and $N=y+\frac{x}{x^2+y^2}$

$$\Rightarrow \frac{\partial N}{\partial x}=0+\left[\frac{x^2y^2-x.2x}{(x^2+y^2)}\right]$$

$$=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

∴ Eqution (1) is exact and its solution is given by

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(treating y as constant) (terms free x in N)

$$\frac{x^2}{2}-Y.\frac{1}{y}\tan^{-1}\frac{1}{y}+\frac{y^2}{2}=0$$

$$x^2+y^2-2\tan^{-1}\frac{x}{y}=2c$$

OR $x^2+y^2=2\tan^{-1}\frac{x}{y}+c_1[\text{where } c_1=2c]$

We have $(2ax + by)y\,dx + (ax + 2by)\,x\,dy = 0$

Here, $M = 2axy + by^2 \Rightarrow \frac{\partial M}{\partial y} = 2ax + 2by$

$N = ax^2 + 2bxy \Rightarrow \frac{\partial N}{\partial x} = 2ax + 2by$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\int (2ax + by^2)\,dx + \int 0 = c$

(treating y as a constant) (There is no term free from x in N)

$ax^2y + by^2x = c$

❑ APPLICATION IN SOLVING PHARMACOKINETIC EQUATIONS

1. Analysis of blood sample
2. Analysis of urine sample

LAPLACE TRANSFORM

Points to be covered in this topic

1. INTRODUCTION
2. PROPERTIES OF LAPLACE TRANSFORM
3. LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS
4. INVERSE LAPLACE TRANSFORMS
5. LAPLACE TRANSFORM OF DERIVATIVES
6. APPLICATION TO SOLVE LINEAR DIFFERENTIAL EQUATIONS
7. APPLICATION IN SOLVING CHEMICAL KINETICS AND PHARMACOKINETICS EQUATIONS

❑ INTRODUCTION

- A transformation is a mathematical device which converts one function into another function

We can write $L\{f(t)\} = F(s)$

Where L is known as Laplace operator

- Laplace transformation of a real valued function f(t) of t for all > 0 is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

❑ PROPERTIES OF LAPLACE TRANSFORMATION

1. Homogeneity: If any constant, then

$$L\{kf(t)\} = \int_0^\infty e^{-st}.kf(t)dt$$

$$= k\int_0^\infty e^{-st}.f(t)dt$$

$$= kL\{f(t)\}$$

$$L\{kf(t)\} = kF(s)$$

2. Additive property

$$\text{if } L\{f(t)\} = F(s) \text{ and } L\{g(t)\} = G(s)$$

$$\text{then } L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\} = F(s) + G(s)$$

Proof: By the definition of LT we have

$$L\{f(t) + g(t)\} = \int_0^\infty e^{-st}[f(t) + g(t)]dt$$

$$L\{f(t) + g(t)\} = \int_0^\infty e^{-st} f(t)\, dt + \int_0^\infty e^{-st} g(t)\, dt$$

$$L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$$

$$L\{f(t) + g(t)\} = F(s) + G(s)$$

3. Liner property

If K_1 and K_2 are constant and

$$L\{f(t)\} = F(s), L\{g(t)\} = G(s)$$

$$\text{Then } L\{K_1 f(t) + K_2 g(t)\} = K_1 L\{f(t)\} + K_2 L\{g(t)\}$$

$$= K_1 F(s) + K_2 G(s)$$

$$LHS = L\{k_1 f(t) + k_2 g(t)\}$$

$$= L\{k_1 f(t)\} + L\{k_2 g(t)\}$$

$$= k_1 L\{f(t)\} + k_2 L\{g(t)\}$$

$$= k_1 F(s) + k_2 G(s)$$

4. First translational property

If $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$

By the definition of LT we have

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)} f(t)\, dt$$

$$L\{e^{at} f(t)\} = F(s-a)$$

5. Change of scale property

If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a} F(s-a)$

By the definition of LT we have

$$L\{f(at)\} = \int_0^\infty e^{-st} e^{at} f(at)\, dt$$

$$= \int_0^\infty e^{-st} f(at)\, dt$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\, du$$

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

❑ LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

f(t)	$L\{f(t)\} = F(s)$
1	$\frac{1}{s}, s > 0$
t^n, n is +ve integer	$\frac{n!}{s^{n+1}}, s > 0$
$t^n, n > -1$	$\frac{\left\lceil \overline{n+1} \right.}{s^{n+1}}, s > 0$
e^{at}	$\frac{1}{s-a}, s > (a)$
e^{-at}	$\frac{1}{s+a}$
sin at	$\frac{a}{s^2 + a^2}, s > 0$
cos at	$\frac{s}{s^2 + a^2}, s > 0$
sin h at	$\frac{a}{s^2 - a^2}, s > (a)$
cosh at	$\frac{s}{s^2 - a^2}, s > (a)$

❑ INVERSE LAPLACE TRANSFORMS

- If L{f(t)} = F(s) is called the inverse of laplace transformation of F(s) and denoted by L^{-1} {f(s)} = f(t)
- Here L^{-1} is known as the inverse of laplace transform operator.

Inverse laplace transform of elementary functions are listed as

f(t)	$L^{-1}\{f(s)\} = F(t)$
$\frac{1}{s}$, s > 0	1
$\frac{1}{s-a}$, s > (a)	e^{at}
$\frac{1}{s^n}$, s > 0	$\frac{t^{n-1}}{\lfloor n-1}$ or $\frac{t^{-1}}{\lceil n}$
$\frac{1}{s^2+a^2}$, s>0	$\frac{1}{2}$ sinat
$\frac{s}{s^2 2a^2}$, s>0	cos at
$\frac{1}{s^2-a^2}$, $s>\|a\|$	$\frac{1}{2}$ sinh at
$\frac{s}{s^2-a^2}$, $s>\|a\|$	cosh at

Properties of inverse LT

(1) Linear property : If k_1 and k_2 are constants and

$L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$, then

$$L^{-1}\{k_1F(s)+k_2G(s)\} = k_1L^{-1}\{F(t)\} + k_2L^{-1}\{G(s)\}$$
$$= k_1f(t)+k_2g(t)$$

proof :By defination of L.T.,ew have,

$$L\{k_1f(t)+k_2g(t)\}=\int_0^\infty e^{-st}\left[k_1f(t)+k_2g(t)\right]dt$$

$$=k_1\int_0^\infty e\text{-}st\ f(t)dt+k_2\int_0^\infty e^{-st}g(t)dt$$

$$=k_1F(s)+k_2G(s)$$

$$k_1f(t)+k_2G(t)=L^{-1}\left[k_1F(s)+k_2G(s)\right]$$

(2) First translation or First shifting proprty:

If $L^{-1}\{F(s)\}=f(t)$,then $L^{-1}\{F(s-a)\}=e^{at}f(t)$.

Proof : we know that,

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

$$\therefore\ F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)dt$$

$$=\int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt$$

$$=L\{e^{at}f(t)\}$$

$$\Rightarrow\ L^{-1}\{F(s-a)\}=e^{at}f(t)$$

(3)second translation or second shifting property:

If $L^{-1}\{F(s)\}=f(t)$,then $L^{-1}\{e^{-as}F(s)\}=g(t)$,when

$$g(t)=\left\{\begin{matrix} f(t-a) & t>a \\ 0 & t>a \end{matrix}\right\}$$

proof : we know that

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

$$\therefore\ e^{-as}F(s)=\int_0^\infty e^{-as}.e^{-st}f(t)dt$$

$$=\int_0^\infty e^{-s(t+a)}f(t)dt \quad \text{put } t+a=u \qquad \text{when } t=0, u=a$$

$$dt=du \qquad \text{when } t\to\infty\, u\to\infty$$

$$e^{-as}F(s)=\int_0^\infty e^{-su}f(u-a)du = 0+\int_0^\infty e^{-su}f(u-a)du$$

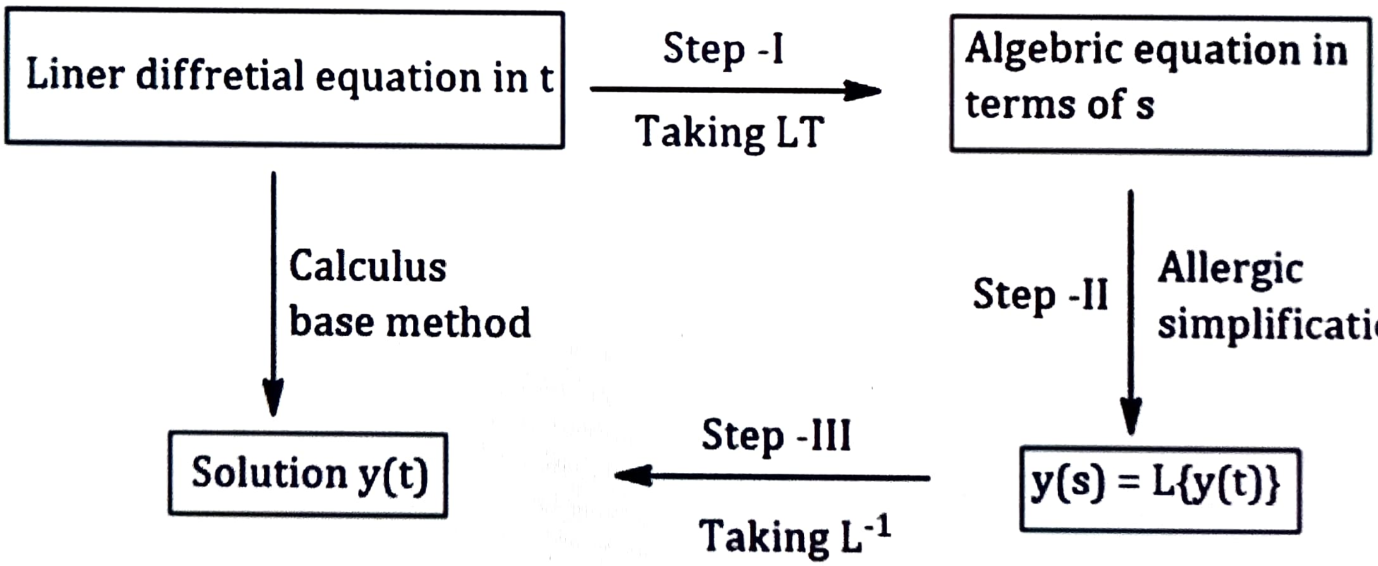
$$=\int_0^a e^{-st}.0\,dt+\int_0^\infty e^{-st}f(t-a)dt$$

$$=\int_0^\infty e^{-st}.g(t)dt, \text{ where, } g(t)=\begin{cases} f(t-a), & t>a \\ 0 & t<a \end{cases}$$

change of scaleproprty: If $L^{-1}\{F(s)\}=f(t)$, then $L^{-1}\{F(as)\}=\frac{1}{a}f\left(\frac{t}{a}\right)$

❑ APPLICATION IN SOLVING CHEMICAL KINETICS AND PHARMACOKINETICS EQUATIONS

- To solve a liner differentiation equation using laplace transform, there are following steps:
- Step – I: Take the LT of both sides of given equation
- Step – II: Simplify algebraically the result to solve $L\{y\} = Y(s)$ in terms of s.
- Step – III: Find inverse laplace transform of $Y(s)$



EXAMPLE

Solve $\frac{dy}{dt} - y = e^t$; $y(0) = 0$

We have $\frac{dy}{dt} - y = e^t$

Taking LT of both the sides, we get

$$L\left\{\frac{dy}{dt}\right\} - L\{y\} = L\{e^t\}$$

$$s\,L\{y\} - y(0) - L\{y\} = \frac{1}{s-1}$$

$$(s-1)\,Y(s) - 0 = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s+1)^2}$$

Taking L^{-1} of both the sides, we get

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$y(t) = e^t.\,L^{-1}\left\{\frac{1}{(s)^2}\right\}$$

$$y(t) = e^t.t$$