DIFFERENTIATION

Points to be covered in this topic

1. INTRODUCTION
2. DERIVATIVE OF A FUNCTION
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4. DERIVATIVE OF A PRODUCT OF A CONSTANT AND A FUNCTION
5. DERIVATIVE OF THE SUM OR DIFFERENCE OF TWO FUNCTIONS
6. DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS (PRODUCT FORMULA)
7. DERIVATIVE OF THE QUOTIENT OF TWO FUNCTIONS (QUOTIENT FORMULA)
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9. SUCCESSIVE DIFFERENTIATION
10. MAXIMA & MINIMA

❑ INTRODUCTION

- Differentiation is a process to finding the derivative for a function at any point
- Let y is a function of x or $y = f(x)$. Here x is independent variable and y is dependent variable. Then the rate of change of dependent variable y with respect to independent variable x is denoted by dy/dx.

❑ DERIVATIVE OF A FUNCTION

$y = f(x)$ ...... (1)

If δy is a small increment in the variable x and δy

is the corresponding change in the variable *y*, then we have

$y + \delta y = f(x + \delta x)$ ....(2)

subtracting (1) from (2), we get

$\delta y = f(x + \delta x) - f(x)$

Dividing by, we get

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

since δx is very small, so we can suppose that $\delta x \to 0$, then

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Or $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad [\delta x = h(let)]$$

Thus, the derivative of a function f(x) can be defined as

$$\frac{d}{dx}[f(x)] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{[right hand derivative]}$$

$$\frac{d}{dx}[f(x)] = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \qquad \text{[left hand derivative]}$$

❑ DERIVATIVE OF A CONSTANT

Let y = constant, then $\frac{dy}{dx} = \frac{d}{dx}(\text{constant}) = 0.$

Proof : Let y = f(x) = k (arbitary constant)

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow \frac{dy}{dx} \lim_{h \to 0} \frac{0}{h} = 0$$

❑ DERIVATIVE OF THE PRODUCT OF A CONSTANT AND A FUNCTION

Let $f(x)$ is a function and k is a arbitary constant,

then $\frac{d}{dx}[k f(x)] = k\frac{d}{dx}f(x)$.

Proof : We have $\frac{d}{dx}[k f(x)] = \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$

$$= \lim_{h\to 0}.\frac{kf(x+h)-kf(x)}{h} = k.\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = k.\frac{d}{dx}f(x).$$

❑ DERIVATIVE OF THE SUM OR DIFFERENCE OF TWO FUNCTIONS

If $f(x)$ and $g(x)$ are two functions of x, then $\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$.

Proof : Let $y = f(x)+g(x)$, then

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[f(x+h)+ g(x+h)-[f(x)+g(x)]]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

In general, $\frac{d}{dx}[f_1(x)+f_2(x)+......+f_n(x)] = \frac{d}{dx}f_1(x)+\frac{d}{dx}f_2(x)+....+\frac{d}{dx}f_n(x)$

Or $\frac{d}{dx}[f_1(x)-f_2(x)-......-f_n(x)] = \frac{d}{dx}f_1(x)-\frac{d}{dx}f_2(x)-....-\frac{d}{dx}f_n(x)$

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$$\frac{d}{dx}[k_1 f_1(x)+k_2 f_2(x)+......+k_n f_n(x)] = k_1\frac{d}{dx}f_1(x)+k_2\frac{d}{dx}f_2(x)+....+k_n\frac{d}{dx}f_n(x)$$

$$\frac{d}{dx}[k_1 f_1(x)-k_2 f_2(x)-......-k_n f_n(x)] = k_1\frac{d}{dx}f_1(x)-k_2\frac{d}{dx}f_2(x)-....-k_n\frac{d}{dx}f_n(x)$$

❑ DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS (PRODUCT FORMULA)

If f(x) and g(x) are two functions then

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Proof : Let y= f(x).g(x), then

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[f(x+h).g(x+h)]-[f(x).g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)+g(x)]-f(x).g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)+g(x)]+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h)-f(x)]}{h}$$

$$= f(x)\lim_{h\to 0}\frac{[g(x+h)-g(x)]}{h} + g(x)\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}$$

$$\therefore \frac{dy}{dx} = f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

Product rule for three functions :

Let $f(x)$, $g(x)$ and $h(x)$ are three functions, then

$$\frac{d}{dx}[f(x)\,g(x)\,h(x)] = \left[\frac{d}{dx}f(x)\right]g(x)\,h(x)+$$

$$f(x)\left[\frac{d}{dx}g(x)\right]h(x)+f(x)g(x)\left[\frac{d}{dx}h(x)\right]$$

❑ DERIVATIVE OF THE QUOTIENT OF TWO FUNCTIONS (QUOTIENT FORMULA)

If $f(x)$ and $g(x)$ and two functions, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

proof : we have,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\left[\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\right] = \lim_{h\to 0}\left[\frac{\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\frac{f(x+h)g(x)-f(x)g(x+h)}{h}}{g(x+h)g(x)}\right]$$

$$=\lim_{h\to 0}\left[\frac{\frac{f(x+h)-f(x)+f(x)g(x)\text{-}f(x)g(x+h)}{h}}{g(x+h)g(x)}\right]$$

$$=\frac{1}{g(x)g(x)}\lim_{h\to 0}\left[\frac{\{f(x+h)-f(x)g(x)+f(x)-f(x)g(x+h)\}}{h}\right]$$

$$=\frac{1}{[g(x)]^2}\left[g(x)\lim_{h\to 0}\left[\frac{\{f(x+h)-f(x)\}}{h}\right]-f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]\right]$$

$$=\frac{1}{[g(x)]^2}\left[g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

$$\therefore \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

❑ DERIVATIVE OF TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLES

1. If y = sin x, then $\frac{dy}{dx}=\frac{d}{dx}(\sin x)=\cos x$

Proof : Let y= f(x) = sin x

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\qquad\left[\because \sin x-\sin y=2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}=\cos x\qquad\left[\because \lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}=1\right]$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\sin x)=\cos x.$$

2. If y = cos x, then $\frac{dy}{dx} = \frac{d}{dx}(\cos x) =$ -sin x.

Proof : Let y= f(x) = cos x

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{-h}{2}\right)\sin\left(\frac{2x+h}{2}\right)}{h} \qquad \left[\because \cos x - \cos y = 2\sin\left(\frac{y-x}{2}\right)\sin\left(\frac{x+y}{2}\right)\right]$$

$$= -\lim_{h \to 0} \sin\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}} \qquad [\because \sin(-x) = -\sin x]$$

$$= -\sin x$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x) = -\sin x.$$

3. If y= tan x ,then $\frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$.

Proof : Let y = f(x) = tan x = $\frac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h\cos(x+h)\cos x} \qquad [\because \sin x\cos y - \cos x\sin y = \sin(x-y)]$$

$$= 1.\frac{1}{\cos x\cos x} = \sec^2 x$$

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x.$$

4. If y= sec x, then $\frac{dy}{dx}=\frac{d}{dx}(\sec x)=\sec x \tan x$

Proof : Let $y = f(x) = \sec x = \frac{1}{\cos x}$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(h+x)-f(x)}{h}=\lim_{h\to 0}\left[\frac{\frac{1}{\cos(x+h)}-\frac{1}{\cos x}}{h}\right]=\lim_{h\to 0}\left[\frac{\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{2\sin\left(\frac{h}{2}\right)\sin\left(\frac{2x+h}{2}\right)}{h\cos(h+x)\cos x}\right]=\lim_{h\to 0}\left[\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]\left[\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(h+x)\cos x}\right]$$

$$=1.\frac{\sin x}{\cos x\cos x}=\tan x\sec x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\cot x)=\tan x\sec x$$

5. If y = cosec x, then $\frac{dy}{dx}=\frac{d}{dx}(\text{cosec } x)=-\text{cosec } x\cot x.$

Proof : Let $y = f(x) = \text{cosec } x = \frac{1}{\sin x}$

$$\frac{dy}{dx}=\lim_{h\to 0} f(x)\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left[\frac{\frac{1}{\sin(x+h)}-\frac{1}{\sin x}}{h}\right]=\lim_{h\to 0}\left[\frac{\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{2\sin\left(-\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h\sin(h+x)\sin x}\right]=\lim_{h\to 0}\left[-\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]\left[\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$-1.\frac{\cos x}{\sin x\sin x}=-\cot x\ \text{cosec } x.$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\text{cosec } x)=-\cot x\ \text{cosec } x.$$

6. if y= cot x,then $\frac{dy}{dx} = \frac{d}{dx}(\cot\ x) = -cosec^2 x.$

proof: Let $= f(x) = \cot x = \frac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}\left[\frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{\cos(x+h)\sin x - \cos x \sin(x+h)}{\sin(x+h)\sin x}}{h}\right]$$

$$\lim_{h \to 0} = \left[\frac{\sin(-h)}{h}\right]\frac{1}{\sin(x+h)\sin x} \quad [\because \sin x \cos x \sin y(x-y)]$$

$$= -1.\frac{1}{\sin x \sin x} = -cosec^2 x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\cot x) = -cosec^2 x.$$

❑ DIFFRENTIATION OF INVERSE TRIGNOMETRIC FUNCTION

$(i)\ \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$(ii)\ \frac{d}{dx}(\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$(iii)\ \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$(iv)\ \frac{d}{dx}(\cot^{-1} x) = \frac{1}{1+x^2}$

$(v)\ \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$(vi)\ \frac{d}{dx}(cosec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

❑ SUCCESSIVE DIFFERENTIATION

- The processes of differentiation of a function again and again is said to be successive differentiation

let y = f(x) is a diffrentiable function of x, then $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots\ldots \frac{d^ny}{dx^n}$

are called 1^{st}, 2^{nd}, 3^{rd}....n^{th} derivative of y respectively.

The successive derivatives of y are also denoted as Dy, D^2y, D^3y.....D^ny

or y1, y2, y3.....yn or f '(x), f "(x), f '''(x).....f^nx)

❑ MAXIMA & MINIMA

- The maxima and minima of a function are the largest or smallest value of a function are the largest or smallest value of the function, either within given range or in the entire domain of a function.